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ANALYSIS OF AN INTEGRATION METHOD FOR RAPIDLY  
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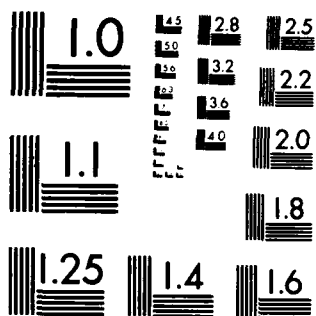
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FOR RAPIDLY OSCILLATING INTEGRANDS

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ANALYSIS OF AN INTEGRATION METHOD FOR RAPIDLY OSCILLATING INTEGRANDS

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ABSTRACT

A collocation method for evaluating integrals of rapidly oscillating functions is analyzed. The method is based on the approximation of the antiderivative of the integrand as the solution of an ordinary differential equation via polynomial collocation. Numerical examples are presented.

AMS (MOS) Subject Classifications: 65D32

Key Words: integration, collocation methods

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# SIGNIFICANCE AND EXPLANATION

Special integration methods have to be used for the integration of highly oscillatory functions. In the present paper we describe such a method and determine the order of convergence.

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# ANALYSIS OF AN INTEGRATION METHOD FOR RAPIDLY OSCILLATING INTEGRANDS

D. Levin\*, L. Reichel and C. Ringhofer

## 1. INTRODUCTION

In [3] a collocation method for computing integrals with rapidly oscillatory integrands was presented, and numerical examples indicated fast convergence. In the present paper, we analyze the method, show convergence rates, and discuss the choice of collocation points.

We are to compute integrals at the form

$$(1.1) \quad I = \int_a^b f(x) e^{iq(x)/\epsilon} dx,$$

where  $-\infty < a < b < \infty$ ,  $i = \sqrt{-1}$ ,  $q(x)$  is a real-valued function,  $f(x)$  is a complex-valued function and  $\epsilon$  is a constant  $0 < \epsilon \ll b - a$ . We assume that  $f(x)$  and  $q(x)$  vary moderately on  $[a, b]$ . Specifically, we assume there are constants  $\delta_1, \delta_2$  such that  $0 < \delta_1 < |\frac{dq}{dx}(x)| < \delta_2$  and  $\delta_1 > \epsilon$  holds. The integration method is based on the observation that if a function  $p(x)$  satisfies

$$(1.2) \quad \frac{d}{dx} (p(x) e^{iq(x)/\epsilon}) = f(x) e^{iq(x)/\epsilon}, \quad a < x < b,$$

then the integral is given by

$$(1.3) \quad I = p(b) e^{iq(b)/\epsilon} - p(a) e^{iq(a)/\epsilon}.$$

From (1.2) it follows that  $p(x)$  is any solution of

$$(1.4) \quad p'(x) + \frac{1}{\epsilon} q'(x) p(x) = f(x), \quad a < x < b.$$

Equation (1.4), we solve by approximating  $p(x)$  by piecewise polynomials, which we determine by collocation. Details as well as convergence results are provided in Section 2. Section 3 contains numerical examples.

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## 2. CONVERGENCE PROPERTIES

Let  $a = z_1 < z_2 < \dots < z_{m+1} = b$  be a partition of  $[a, b]$ , and for notational convenience assume that all  $z_j$  are equidistant with

$$(2.1) \quad h := \frac{b-a}{m} = z_{j+1} - z_j, \quad j = 1(1)m.$$

and  $h\delta_1 > \epsilon$ . On each subinterval  $[z_{j+1}, z_j]$ , we approximate  $p(x)$  by a  $k$ -th degree polynomial, which we determine by collocation in  $k+1$  points. This gives us a convergence order of  $O(\epsilon^2 h^{k-1})$ , and as example 2.1 below shows, this result is sharp. From lemma 2.1 below it follows that  $|I| = O(\epsilon)$ , why the relative accuracy of the method is  $O(\epsilon h^{k-1})$ . We next describe the collocation method. Choose  $k+1$  reference points  $w_j$ ,  $0 = w_0 < w_1 < \dots < w_k = 1$ . The collocation points  $w_s^j$  in  $[z_{j+1}, z_j]$ , we obtain by the linear map

$$(2.2) \quad w_s^j = z_j + h w_s, \quad s = 0(1)k, \quad j = 1(1)m.$$

Let  $\ell_{nj}(x)$  be the Lagrange polynomials defined by  $w_s^j$ ,  $s = 0(1)k$ , i.e.

$$(2.3) \quad \ell_{n,j}(x) := \prod_{\substack{s=0 \\ s \neq n}}^k \frac{x - w_s^j}{w_n^j - w_s^j}, \quad n = 0(1)k.$$

Determine the polynomial

$$(2.4) \quad p_j(x) = \sum_{n=0}^k a_{nj} \ell_{nj}(x)$$

from

$$(2.5) \quad p_j'(x) + \frac{1}{\epsilon} q'(x) p_j(x) = f(x), \quad x = w_0^j, w_1^j, \dots, w_k^j.$$

The computed  $p_j(x)$  approximates  $p(x)$  on  $[z_j, z_{j+1}]$ . Analogously to (1.3)

$$(2.6) \quad I_j^k := p_j(z_{j+1}) e^{iq(z_{j+1})/\epsilon} - p_j(z_j) e^{iq(z_j)/\epsilon}$$

is an approximation of  $\int_{z_j}^{z_{j+1}} f(x) e^{iq(x)/\epsilon} dx$ , and

$$(2.7) \quad I^k := \sum_{j=1}^{m+1} I_j^k$$

approximates  $I$ .

Theorem 2.1

Let  $I, I^k, h, p_j$  etc. be defined by (1.1) and (2.1)-(2.7). Then

$$|I - I^k| < c\epsilon^2 h^{k-1},$$

where  $c$  is a constant independent of  $\epsilon$  and  $h$ .

To prove theorem (2.1) we first need some additional lemmas: As we will see it is essential for our method that there exists a sufficiently smooth solution of (2.5), i.e. that there exists a solution which has sufficiently many derivatives bounded uniformly as  $\epsilon$  goes to zero. This is shown in

Lemma 2.1

Let  $f(x) \in C^{2k-1}[z_j, z_{j+1}]$  and  $q(x) \in C^{2k}[z_j, z_{j+1}]$ . Then there is a solution  $v(x)$

$$(2.8) \quad v'(x) + \frac{1}{\epsilon} q'(x)v(x) = f(x), \quad z_j < x < z_{j+1},$$

which satisfies

$$(2.9) \quad \left| \frac{d^s v}{dx^s}(x) \right| < c\epsilon \quad \text{on } [z_j, z_{j+1}], \quad s = 0(1)k$$

where  $c$  is a constant independent of  $\epsilon$ .

Proof. Let

$$v_k(x) := \sum_{s=1}^k v_s(x) \epsilon^s,$$

$$v_1(x) := \frac{f(x)}{iq'(x)}, \quad v_s(x) := -\frac{v_{s-1}'(x)}{iq'(x)}, \quad s = 2(1)k.$$

Then  $v_k$  satisfies

$$v_k'(x) + \frac{1}{\epsilon} q'(x)v_k(x) = f(x) + \epsilon^k v_k'(x)$$

and

$$\frac{d^s v_k}{dx^s}(x) = O(\epsilon), \quad z_j < x < z_{j+1}, \quad s = 0(1)k.$$

Now

$$v(x) := v_k(z_j) e^{iq(z_j)/\epsilon} e^{-iq(x)/\epsilon} + e^{-iq(x)/\epsilon} \int_{z_j}^x e^{iq(t)/\epsilon} f(t) dt$$



solves (2.8). Introduce  $e(x) := v(x) - V_K(x)$ . Then

$$e'(x) + \frac{1}{\epsilon} q'(x)e(x) = v'_K(x)\epsilon^k, \quad e(z_j) = 0.$$

Then  $e(x)$  can be written as

$$(2.10) \quad e(x) = \epsilon^K e^{-iq(x)/\epsilon} \int_{z_j}^x e^{iq(t)/\epsilon} v'_K(t) dt.$$

From

$$\int_{z_j}^x e^{iq(t)/\epsilon} v'_K(t) dt = O(\epsilon), \quad z_j < x < z_{j+1},$$

it follows that  $e(x) = O(\epsilon^{K+1})$  on  $[z_j, z_{j+1}]$ . Repeated differentiation of (2.11) shows that  $\frac{d^s e}{dx^s}(x)$  is at most  $O(\epsilon^{K+1-s})$ ,  $s = O(1)K$ . By construction  $V_K(x)\epsilon^{-1}$  has  $K$  derivatives bounded independently of  $\epsilon$ . The lemma follows from  $v(x) = V_K(x) + e(x)$ .

#### Remark 2.1

The smoothness requirements on  $f(x)$  and  $q(x)$  are only sufficient. It is simple to construct functions where  $v(x)$  has more continuous derivatives than  $f(x)$  and  $q(x)$ .

See section 3.

Next we estimate the error in  $I_j^k$  (see (2.6)) on each subinterval  $[z_j, z_{j+1}]$ . Let  $v(x)$  solve (2.8). Then

$$(2.11) \quad \begin{aligned} & \left| \int_{z_j}^{z_{j+1}} f(x) e^{iq(x)/\epsilon} dx - I_j^k \right| = \\ & = |(v(z_{j+1}) - p_j(z_{j+1})) e^{iq(z_{j+1})/\epsilon} - (v(z_j) - p_j(z_j)) e^{iq(z_j)/\epsilon}| < \\ & < |v(z_{j+1}) - p_j(z_{j+1})| + |v(z_j) - p_j(z_j)|. \end{aligned}$$

Hence, it suffices to bound  $v(x) - p_j(x)$  of  $z_j$  and  $z_{j+1}$ .

#### Lemma 2.2

Let  $v(x)$  and  $p_j(x)$  have the same meaning as in (2.11). At the collocation points  $w_s^j$ ,  $s = O(1)k$ , in  $[z_j, z_{j+1}]$ , (see (2.2))

$$(2.12) \quad |v(w_s^j) - p_j(w_s^j)| < \epsilon c \max_{0 \leq s \leq k} |r_s| ,$$

holds where  $c$  is a constant independent of  $\epsilon$  and  $h$ , and the  $r_s$  are defined as follows: let  $p_j(x)$  be the polynomial of degree  $< k$  satisfying  $\hat{p}_j(w_s^j) = v(w_s^j)$ ,  $s = 0(1)k$ . We define  $r_s$  as the residuals

$$(2.13) \quad r_s := v'(w_s^j) - \hat{p}_j'(w_s^j), \quad s = 0(1)k .$$

Proof

With the representation (2.4) of  $p_j(x)$ , the collocation equations (2.5) become

$$(2.14) \quad (L + \frac{1}{\epsilon} D) \underline{a} = \underline{f} ,$$

where

$$D := \text{diag}(iq'(w_0^j), iq'(w_1^j), \dots, iq'(w_k^j))$$

$$L := \begin{pmatrix} l_{0j}'(w_0^j) & \dots & l_{kj}'(w_0^j) \\ \vdots & & \vdots \\ l_{0j}'(w_k^j) & \dots & l_{kj}'(w_k^j) \end{pmatrix}$$

$$\underline{f} := (f(w_0^j), \dots, f(w_k^j))^T$$

$$\underline{a} := (a_{0j}, \dots, a_{kj})^T$$

Let  $v(x)$  be any solution of (2.8), and let  $\underline{v} := (v(w_0^j), \dots, v(w_k^j))^T$ . We write

$$\hat{p}_j'(w_s^j) + \frac{1}{\epsilon} q'(w_s^j) \hat{p}_j(w_s^j) = f(w_s^j) + r_s, \quad s = 0(1)k, \text{ as}$$

$$(2.15) \quad (L + \frac{1}{\epsilon} D) \underline{v} = \underline{f} + \underline{r} ,$$

where  $\underline{r} = (r_0, r_1, \dots, r_k)^T$ . Since the elements of  $L$  are  $O(\frac{1}{h})$ ,  $|q'| > \delta_1$  and  $\epsilon < \delta, h$

the matrix  $L + \frac{1}{\epsilon} D$  is invertible, and

$$\underline{v} - \underline{a} = \epsilon(\epsilon L + D)^{-1} \underline{r} = \epsilon D^{-1} \sum_{s=0}^{\infty} (-LD^{-1}\epsilon)^s \underline{r} .$$

This proves the lemma.

The next lemma bounds  $\underline{r}$ . The result is well-known and a simple proof by Rolle's theorem is omitted.

Lemma 2.3

Let  $p(x)$  be the polynomial of degree  $\leq k$  that interpolates  $g(x) \in C^{k+1}[z_j, z_{j+1}]$  at points  $w_0^j, w_1^j, \dots, w_k^j$ . Then for any  $x \in [z_j, z_{j+1}]$ ,

$$|\hat{p}'(x) - g'(x)| \leq ch^k \max_{z_j \leq x \leq z_{j+1}} \left| \frac{d^{k+1}g(x)}{dx^{k+1}} \right|,$$

where  $c$  is a constant independent of  $h$ .

We are now in a position to give an estimate for the integration error for one subinterval.

Theorem 2.2

Let  $I_j^k$  be defined by (2.6), and assume that  $f(x)$  and  $g(x)$  satisfy the smoothness requirements of lemma 2.1. Then

$$(2.16) \quad |I_j^k - \int_{z_j}^{z_{j+1}} f(x) e^{iq(x)/\epsilon} dx| \leq c\epsilon^2 h^k,$$

where  $c$  is a constant independent of  $\epsilon$  and  $h$ .

Proof. Taking the smooth solution of lemma 2.1 with  $\kappa = k+1$ , its  $k+1$ st derivative is continuous and  $O(\epsilon)$ . By lemma 2.3,  $r_g$  introduced in lemma 2.2 satisfies

$\max_{0 \leq s \leq k} |r_g^s| \leq c\epsilon h^k$  for some constant  $c$  independent of  $\epsilon$  and  $h$ . Therefore the theorem follows by (2.11) and (2.12).

The proof of theorem 2.1 is now trivial. It follows from theorem 2.2 and summation over all  $m = (b-a)/h$  subintervals. The following example shows that the error estimate of theorem 2.1 is sharp if  $\epsilon \leq h\delta_1$ .

Ex. 2.1 Consider the integral

$$I = \int_0^1 e^x e^{ix/\epsilon} dx.$$

Divide  $[0,1]$  into  $m$  subintervals  $[z_j, z_{j+1}]$  of length  $h := 1/m$ , i.e.  $z_j = h_j$ .

$j = 0(1)m$ . For future reference, we write  $I$  as

$$(2.17) \quad I = \sum_{j=0}^{m-1} \int_{z_j}^{z_{j+1}} e^x e^{ix/\epsilon} dx = \frac{\epsilon}{1+\epsilon} (e^h e^{ih/\epsilon} - 1) \sum_{j=0}^{m-1} e^{z_j} e^{iz_j/\epsilon}.$$

We apply our integration method with  $k = 1$  on each subinterval  $[z_j, z_{j+1}]$ . We determine a linear function

$$p_j(x) = d_{0j} \frac{z_{j+1} - x}{h} + d_{1j} \frac{x - z_j}{h}$$

which satisfies

$$p_j'(z_s) = \frac{1}{\epsilon} p_j(z_s) = e^{z_s}, \quad s = j, j+1.$$

This determines

$$I^1 := \sum_{j=0}^{m-1} (e^{iz_{j+1}/\epsilon} d_{1j} - e^{iz_j/\epsilon} d_{0j}),$$

which after some simplification can be written as

$$(2.18) \quad I^1 = \left[ \frac{\epsilon}{1} (e^h e^{ih/\epsilon} - 1) + \epsilon^2 \frac{e^h - 1}{h} (e^{ih/\epsilon} - 1) \right] \sum_{j=0}^{m-1} e^{z_j} e^{iz_j/\epsilon}.$$

with (2.17) this gives

$$(2.19) \quad I^1 - I = \left[ \frac{-\epsilon}{1-i\epsilon} (e^h e^{ih/\epsilon} - 1) + \epsilon^2 \frac{e^h - 1}{h} (e^{ih/\epsilon} - 1) \right] \sum_{j=0}^{m-1} e^{z_j} e^{iz_j/\epsilon} \\ = [0(\epsilon^3) + \epsilon^2 o(h)] \sum_{j=0}^{m-1} e^{z_j} e^{iz_j/\epsilon}.$$

The sum in (2.19) is bounded by  $O(1/h)$ , and therefore

$$|I^1 - I| \leq O(\epsilon^2) + O(\epsilon^3/h).$$

From (2.19) it is obvious that theorem 2.1 gives a bound for the decay of the error, while the error itself might not tend to zero smoothly as  $h \rightarrow 0$  or  $\epsilon \rightarrow 0$ . This is also illustrated by the numerical examples of section 3.

We conclude this section with a suggestion for collocation points. In the proofs we have only required that the collocation points are distinct on each subinterval  $[z_j, z_{j+1}]$  and that the end points of each subinterval are collocation points. While the allocation of collocation points does not affect the rate of convergence  $I^k + I$  as  $h \rightarrow 0$  or as  $\epsilon \rightarrow 0$ , the allocation does affect the constant  $c$  in (2.16) and in theorem 2.1. Let  $D$  denote the differentiation operator, and write (2.8) symbolically as

$$(I + \frac{\epsilon}{iq'} D)v = \frac{\epsilon}{iq'} f$$

or equivalently, if  $\epsilon$  is sufficiently small and  $f$  is sufficiently smooth,

$$(2.20) \quad v = (I + \frac{\epsilon}{iq'} D)^{-1} \frac{\epsilon}{iq'} f = \frac{\epsilon}{iq'} f + \frac{\epsilon^3}{(q')^2} f' + \frac{\epsilon^3}{i(q')^3} f'' + \dots$$

This shows that the polynomial  $p_j(x)$  we determine by (2.5) is close to the  $k$ th degree polynomial  $p_j^I(x)$  which interpolates  $\frac{\epsilon}{iq'} f$  at  $w_s^j$ ,  $s = 0(1)k$ , and it suggests the use of collocation points  $w_s^j$ , which are good for the interpolation problem

$p_j^I(w_s^j) = \frac{\epsilon}{i} f(w_s^j)(q'(w_s^j))^{-1}$ ,  $s = 0(1)k$ . If we select the  $w_s^j$  as the extended Chebyshev points

$$(2.21) \quad w_\ell^j := \frac{1}{2} (z_j + z_{j+1} - (z_j - z_{j+1}) \frac{\cos((2\ell + 1)\pi/(2k + 2))}{\cos(\pi/(2k + 2))}), \quad \ell = 0(1)k,$$

then  $p_j^I(x)$  will be close to the best polynomial approximant to  $\frac{\epsilon}{i} f(x)(q'(x))^{-1}$ , see de Boor [1], ch. 2 or Brutman [2]. The examples of section 3 confirm that the extended Chebyshev points also are good for solving the collocation problem (2.5).

### 3. NUMERICAL EXAMPLES

In all examples the interval is  $[0,1]$ . The  $m$  subintervals  $[z_j, z_{j+1}]$  are defined by  $z_j = h_j$ ,  $h = \frac{1}{m}$ ,  $j = 0(1)m$ .  $k$  denotes the degree of the polynomial  $p_j(x)$ . If nothing else stated, the collocation points  $\{w_s^j\}_{s=0}^k$  are the extended Chebyshev points (2.21). All computations were done on a VAX 11/260 in double precision arithmetic, i.e. with 12 significant digits.

Ex. 3.1 Let  $q(x) := x + x^2$  and

$$f(x) := 1 \frac{1 + 2x}{(x - 0.5)^2 + 0.1} - \frac{2x - 1}{((x - 0.5)^2 + 0.1)^2} \varepsilon.$$

Then (1.4) is solved by  $p(x) = \varepsilon((x - 0.5)^2 + 0.1)^{-1}$ . The table shows the dependence of  $|I - I^k|$  on  $h$  and on  $\varepsilon$ .

| $h$  | $k$ | $\epsilon$ | $I^k$ (rounded)                                 | $ I - I^k $          |
|------|-----|------------|---|----------------------|
| 1    | 5   | $10^{-3}$  | $-3.9093 \cdot 10^{-3} + 2.6642 \cdot 10^{-3}i$ | $7.3 \cdot 10^{-6}$  |
| 1/2  | 5   | $10^{-3}$  | $-3.9070 \cdot 10^{-3} + 2.6573 \cdot 10^{-3}i$ | $1.4 \cdot 10^{-6}$  |
| 1/4  | 5   | $10^{-3}$  | "   | $5.0 \cdot 10^{-8}$  |
| 1/8  | 5   | $10^{-3}$  | "   | $4.7 \cdot 10^{-9}$  |
| 1/16 | 5   | $10^{-3}$  | "   | $1.7 \cdot 10^{-10}$ |
| 1    | 5   | $10^{-6}$  | $-6.9997 \cdot 10^{-7} - 1.8735 \cdot 10^{-6}i$ | $1.0 \cdot 10^{-11}$ |
| 1/2  | 5   | $10^{-6}$  | "   | $1.5 \cdot 10^{-12}$ |
| 1/4  | 5   | $10^{-6}$  | "   | $8.8 \cdot 10^{-14}$ |
| 1/8  | 5   | $10^{-6}$  | "   | $3.7 \cdot 10^{-15}$ |
| 1/16 | 5   | $10^{-6}$  | "   | $2.3 \cdot 10^{-16}$ |
| 1/2  | 20  | $10^{-3}$  | $-3.9070 \cdot 10^{-3} + 1.6573 \cdot 10^{-3}i$ | $2.9 \cdot 10^{-14}$ |
| 1/2  | 20  | $10^{-3}$  | $-6.9997 \cdot 10^{-7} - 1.8735 \cdot 10^{-6}i$ | $1.4 \cdot 10^{-19}$ |
| 1/2  | 20* | $10^{-3}$  | $-3.9070 \cdot 10^{-3} + 2.6573 \cdot 10^{-3}i$ | $1.3 \cdot 10^{-11}$ |

\*equidistant collocation point

We note that high accuracy can be obtained not only by decreasing  $h$ , but also by increasing  $k$ . The latter strategy can be competitive since the strong diagonal dominance of system (2.14) allows iterative solution in  $O(k^2)$  operations.

Ex. 3.2 Let  $g(x) := x + x^2$  and  $f(x) := (1+2x)(x - \frac{1}{\sqrt{2}})^2 \text{sign}(x - \frac{1}{\sqrt{2}}) - 2(x - \frac{1}{\sqrt{2}})\epsilon$ . Then  $p(x) := (x - \frac{1}{\sqrt{2}})^2 \text{sign}(x - \frac{1}{\sqrt{2}})$  solves (1.4).

| h    | k  | $\epsilon$ | $I^k$ (rounded)                                | $ I - I^k $          |
|------|----|------------|--|----------------------|
| 1/8  | 2  | $10^{-3}$  | $4.6847 \cdot 10^{-4} + 7.9750 \cdot 10^{-5}i$ | $3.63 \cdot 10^{-8}$ |
| 1/16 | 2  | $10^{-3}$  | $4.6849 \cdot 10^{-4} + 7.9770 \cdot 10^{-5}i$ | $1.85 \cdot 10^{-8}$ |
| 1/4  | 8  | $10^{-3}$  | $4.6848 \cdot 10^{-4} + 7.9788 \cdot 10^{-5}i$ | $4.47 \cdot 10^{-9}$ |
| 1/8  | 8  | $10^{-3}$  | $4.6848 \cdot 10^{-4} + 7.9783 \cdot 10^{-5}i$ | $2.74 \cdot 10^{-9}$ |
| 1/4  | 8* | $10^{-3}$  | $4.6849 \cdot 10^{-4} + 7.9809 \cdot 10^{-5}i$ | $2.9 \cdot 10^{-8}$  |

\*equidistant collocation points.

The example shows the robustness of the method.

The last example shows that the conditions of lemma 2.1 are not necessary.

Ex. 3.3 Let  $q(x) := |x - \frac{1}{\sqrt{2}}|$  and define  $q'(\frac{1}{\sqrt{2}}) = 0$ . Let

$$f(x) := \frac{q'(x)i}{(x - 0.5)^2 + 0.1} - \frac{2x - 1}{((x - 0.5)^2 + 0.1)^2} \epsilon.$$

Then  $p(x)$  is the same as in Example 3.1.

| h    | k | $\epsilon$ | $I^k$ (rounded)                                | $ I - I^k $          |
|------|---|------------|--|----------------------|
| 1/8  | 5 | $10^{-3}$  | $6.3105 \cdot 10^{-4} - 1.1924 \cdot 10^{-3}i$ | $6.0 \cdot 10^{-9}$  |
| 1/16 | 5 | $10^{-3}$  | "  | $9.3 \cdot 10^{-11}$ |



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